

# String Amplitudes and N=2, $d = 4$ Prepotential in Heterotic $K3 \times T^2$ Compactifications

K. Förger<sup>1</sup> and S. Stieberger<sup>2</sup>

<sup>1</sup>*Sektion Physik*

*Universität München*

*Theresienstraße 37*

*D-80333 München, FRG*

<sup>2</sup>*Institut de Physique Théorique*

*Université de Neuchâtel*

*CH-2000 Neuchâtel, SWITZERLAND*

For the gauge couplings, which arise after toroidal compactification of six-dimensional heterotic N=1 string theories from the  $T^2$  torus, we calculate their one-loop corrections. This is performed by considering string amplitudes involving two gauge fields and moduli fields. We compare our results with the equations following from N=2 special geometry and the underlying prepotential of the theory. Moreover we find relations between derivatives of the N=2,  $d = 4$  prepotential and world-sheet  $\tau$ -integrals which appear in various string amplitudes of any  $T^2$ -compactification.

## 1. Introduction

The vector multiplet sector of N=2 supergravity in four dimensions is governed by a holomorphic function, namely the prepotential  $F$  [1]. Up to second order derivatives the Lagrangian is constructed from it. The Kähler metric and the gauge couplings are expressible by derivatives of this prepotential. Effective N=2 supergravity theories in four dimensions arise e.g. after compactifications of heterotic string theories on  $K3 \times T^2$  or type IIA or type IIB on a Calabi–Yau (CY) threefold. In the following we will concentrate on the first vacuum. In addition to the heterotic gauge group, which depends on the specific instanton embeddings and may be –in certain cases– completely Higgsed away, a  $K3 \times T^2$  heterotic string compactification always possesses the  $U(1)_+^2 \times U(1)_-^2$  internal gauge symmetry. The first factor comes from the internal graviton and the last factor arises from the compactification of the six–dimensional tensor multiplet which describes the heterotic dilaton in six dimensions. The gauge fields of the left  $U(1)_L$ ’s belong to vector multiplets whose scalars are the  $T$  and  $U$  moduli of the torus  $T^2$ . Their moduli space is described by special geometry [2][3]. Besides, there is a vector multiplet for the heterotic dilaton and the graviphoton, whereas the  $K3$  moduli and the gauge bundle deformations come in scalars of hyper multiplets.

At special points in the  $T, U$  moduli space, the  $U(1)_L^2$  may be enhanced to  $SU(2)_L \times U(1)_L$  for  $T = U$ , to  $SU(2)_L^2$  for  $T = U = i$  and  $SU(3)_L$  at  $T = U = \rho$  where  $\rho = e^{2\pi i/3}$  [4]. Logarithmic singularities in the effective string action appear at these special points. This effect should be seen e.g. in the one–loop gauge couplings, where heavy strings have been integrated out. Modes, which have been integrated out and become light at these special points in the moduli space are responsible for these singularities. These one–loop couplings can be expressed by the perturbative prepotential. Therefore, a calculation of one–loop gauge couplings gives information about the prepotential (and vice versa). Until now<sup>1</sup>, no threshold calculation has been undertaken for gauge groups, where the modulus and the gauge boson under consideration sit in the same vector multiplet. In the following we focus on the perturbative prepotential of the vector multiplets.

One–loop gauge threshold corrections are very important quantities for at least three aspects of string theory: They play an important rôle in constructing consistent, i.e. anomaly free, effective string actions [6], they are sensitive quantities for heterotic–typeII string

---

<sup>1</sup> Except [5].

duality tests [7][8][9] and shed light on the perturbative and hopefully also on the non-perturbative part of the prepotential [10]. In this paper we will focus on the last issue: We calculate threshold corrections to the  $U(1)_L^2$  gauge bosons which are given as certain  $\tau$ -integrals. The latter we can express in terms of the prepotential and its derivatives. This not only gives a check of the framework of N=2 supergravity emerging for heterotic  $K3 \times T^2$  string compactifications, which tells us, how these corrections have to be expressed by the prepotential, but it gives relations between string amplitudes given as world-sheet torus integrals and the prepotential. Such relations are important for understanding the structure of string amplitudes in any dimensions and its relation to a function, which in  $d = 4$  is the N=2 prepotential.

The organization of the present paper is as follows: In section 2 we briefly review some facts about N=2 supergravity in light of the gauge couplings. Section 3 is devoted to a string derivation of the  $U(1)_L^2$ -gauge couplings by calculating the relevant string amplitudes. In section 4 we find various relations between world-sheet  $\tau$ -integrals and combinations of the prepotential and its derivatives. These relations allow us to express the string amplitudes in terms of the prepotential which is the appropriate form to compare with the supergravity formulae of section 2. In section 5 we trace back the origin of the gauge couplings to six dimensions via the elliptic genus. Section 6 gives a summary of our results and some concluding remarks. All technical details for the  $\tau$ -integrations are presented in the appendix.

## 2. N=2 supergravity and effective string theory

For N=2,  $d = 4$  heterotic string vacua arising from  $K3 \times T^2$  compactifications, the dilaton field  $S$ , the  $T$  and  $U$  moduli describing the torus  $T^2$  (and possible Wilson lines) are scalar fields of N=2 vector multiplets [11][12]. The absence of couplings between scalars of vector multiplets and scalars of hyper multiplets (describing e.g. the K3 moduli) allows one to study the two moduli spaces separately [3]. Since the dilaton field  $S$  comes in a vector multiplet, the prepotential describing the gauge sector may receive space-time perturbative and non-perturbative corrections in contrast to the hyper multiplet moduli space, which does not get any perturbative corrections. However in N=2 supergravity one expects a non-renormalization theorem following from chiral superspace integrals which prohibits higher than one-loop corrections to the prepotential. Besides, in heterotic string theories, the dilaton obeys a continuous Peccei-Quinn symmetry to all orders in perturbation theory

which also forbids higher than one-loop corrections. Therefore, the only perturbative correction to the tree-level prepotential comes at one-loop, summarized by a function  $f$ . For the prepotential describing the  $S, T$  and  $U$  moduli space of a heterotic  $K3 \times T^2$  compactification one has (neglecting non-perturbative corrections) [12][13]

$$F(X) = \frac{X^1 X^2 X^3}{X^0} + (X^0)^2 f\left(\frac{X^2}{X^0}, \frac{X^3}{X^0}\right), \quad (2.1)$$

with the unconstrained vector multiplets  $X^I$ ,  $I = 0, \dots, 3$ . The function  $f$  is much constrained by the perturbative duality group  $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U \times \mathbf{Z}_2^{T \leftrightarrow U}$ . The field  $X^0$  accounts for the additional vector multiplet of the graviphoton. The gauge kinetic part of the Lagrangian is

$$\mathcal{L}_{\text{gauge}} = -\frac{i}{4}(\mathcal{N}_{IJ} - \bar{\mathcal{N}}_{IJ})F_{\mu\nu}^I F^{\mu\nu J} + \frac{1}{4}(\mathcal{N}_{IJ} + \bar{\mathcal{N}}_{IJ})F_{\mu\nu}^I \tilde{F}^{\mu\nu J}, \quad (2.2)$$

where the gauge couplings are then expressed in terms of  $\mathcal{N}_{IJ}$

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i \frac{\text{Im} F_{IK} \text{Im} F_{JL} X^K X^L}{\text{Im} F_{MN} X^M X^N} \quad (2.3)$$

via

$$g_{IJ}^{-2} = \mathcal{N}_{IJ} - \bar{\mathcal{N}}_{IJ}. \quad (2.4)$$

The scalar partner (and spinor) of the graviphoton is gauge fixed in super Poincaré gravity. With the standard choice for the scalar fields of the vector multiplets

$$S = \frac{X^1}{X^0}, \quad T = \frac{X^2}{X^0}, \quad U = \frac{X^3}{X^0}, \quad (2.5)$$

we derive from (2.3) and (2.4) for the effective gauge couplings up to order  $\mathcal{O}(f^2, (\partial f)^2/(S - \bar{S}))$

$$\begin{aligned} g_{22}^{-2} &= \frac{(S - \bar{S})(U - \bar{U})}{(T - \bar{T})} - \frac{1}{4}[\partial_T D_T f - \partial_{\bar{T}} D_{\bar{T}} \bar{f}] - \frac{1}{4} \frac{(U - \bar{U})^2}{(T - \bar{T})^2} [\partial_U D_U f - \partial_{\bar{U}} D_{\bar{U}} \bar{f}] \\ &\quad + \frac{1}{2} \frac{(U - \bar{U})}{(T - \bar{T})} [\partial_T \partial_U f - \partial_{\bar{T}} \partial_{\bar{U}} \bar{f}] + \mathcal{O}\left(\frac{f^2, (\partial f)^2}{S - \bar{S}}\right) \\ g_{23}^{-2} &= (S - \bar{S}) - \frac{1}{2} [D_T D_U f - D_{\bar{T}} D_{\bar{U}} \bar{f}] + \frac{1}{4} \frac{(T - \bar{T})}{(U - \bar{U})} [\partial_T D_T f - \partial_{\bar{T}} D_{\bar{T}} \bar{f}] \\ &\quad + \frac{1}{4} \frac{(U - \bar{U})}{(T - \bar{T})} [\partial_U D_U f - \partial_{\bar{U}} D_{\bar{U}} \bar{f}] + \mathcal{O}\left(\frac{f^2, (\partial f)^2}{S - \bar{S}}\right), \end{aligned} \quad (2.6)$$

where the covariant derivatives are  $D_T = \partial_T - \frac{2}{(T-\bar{T})}$  and  $D_{\bar{T}} = \partial_{\bar{T}} + \frac{2}{(T-\bar{T})}$  and the indices  $T, U$  correspond to  $I = 2, 3$  respectively. Of course, there is no dilaton dependence at one-loop. The expression for  $g_{33}^{-2}$  we simply deduce from  $g_{22}^{-2}$  by exchanging  $T$  and  $U$ . The couplings in (2.6) are related via  $\frac{(T-\bar{T})}{(U-\bar{U})}g_{22}^{-2} + g_{23}^{-2} = 2(S-\bar{S}) - \frac{1}{2} \left( D_T D_U f - \partial_T \partial_U f - hc \right)$ .

In the interpretation of (2.6) one occurs a puzzle: Along the explanations of above the gauge-couplings should not receive higher than one-loop contributions, i.e. powers in  $1/(S-\bar{S})$  must not appear. Nonetheless, it is the dilaton independent part of (2.6), which is relevant for the string one-loop calculations in the next section. This is in precise analogy of [13], where an expansion like in (2.6) has been performed for the one-loop Kähler metric. All the same, for completeness, let us mention the solution to that puzzle in view of the gauge-couplings (2.3). The symplectic transformation  $(X^I, F_J) \rightarrow (\hat{X}^I, \hat{F}_J)$  [12]

$$\begin{aligned} \hat{X}^1 &= F_1, \quad \hat{F}_1 = -X^1, \\ \hat{X}^I &= X^I, \quad \hat{F}_I = F_I, \quad I \neq 1 \end{aligned} \quad (2.7)$$

changes the metric from  $\mathcal{N}$  to  $\hat{\mathcal{N}}$  with

$$\hat{\mathcal{N}} = \begin{pmatrix} \mathcal{N}_{00} - \frac{\mathcal{N}_{01}\mathcal{N}_{10}}{\mathcal{N}_{11}} & \frac{\mathcal{N}_{01}}{\mathcal{N}_{11}} & \mathcal{N}_{02} - \frac{\mathcal{N}_{01}\mathcal{N}_{12}}{\mathcal{N}_{11}} & \mathcal{N}_{03} - \frac{\mathcal{N}_{01}\mathcal{N}_{13}}{\mathcal{N}_{11}} \\ \frac{\mathcal{N}_{10}}{\mathcal{N}_{11}} & -\frac{1}{\mathcal{N}_{11}} & \frac{\mathcal{N}_{12}}{\mathcal{N}_{11}} & \frac{\mathcal{N}_{13}}{\mathcal{N}_{11}} \\ \mathcal{N}_{20} - \frac{\mathcal{N}_{10}\mathcal{N}_{21}}{\mathcal{N}_{11}} & \frac{\mathcal{N}_{21}}{\mathcal{N}_{11}} & \mathcal{N}_{22} - \frac{\mathcal{N}_{12}\mathcal{N}_{21}}{\mathcal{N}_{11}} & \mathcal{N}_{23} - \frac{\mathcal{N}_{13}\mathcal{N}_{21}}{\mathcal{N}_{11}} \\ \mathcal{N}_{30} - \frac{\mathcal{N}_{10}\mathcal{N}_{31}}{\mathcal{N}_{11}} & \frac{\mathcal{N}_{31}}{\mathcal{N}_{11}} & \mathcal{N}_{32} - \frac{\mathcal{N}_{12}\mathcal{N}_{31}}{\mathcal{N}_{11}} & \mathcal{N}_{34} - \frac{\mathcal{N}_{13}\mathcal{N}_{31}}{\mathcal{N}_{11}} \end{pmatrix}. \quad (2.8)$$

It can be verified that in this new basis (2.8), all gauge couplings involve neither powers of  $1/(S-\bar{S})$  nor higher orders in  $f$  or its derivatives. This is just an effect of a rearrangement of all couplings  $\mathcal{N}_{IJ}$  (2.3) in (2.8). E.g. for  $\hat{g}_{22}^{-2} \equiv \hat{\mathcal{N}}_{22} - \bar{\hat{\mathcal{N}}}_{22} = 2i\text{Im} \left( \mathcal{N}_{22} - \frac{\mathcal{N}_{12}^2}{\mathcal{N}_{11}} \right)$  one determines

$$\begin{aligned} \hat{g}_{22}^{-2} &= -4(\tilde{S} - \bar{\tilde{S}}) \frac{|U|^2}{(T-\bar{T})(U-\bar{U})} \\ &\quad - \frac{1}{(U-\bar{U})^2} \left[ \bar{U}^2 \partial_T D_T f - U^2 \partial_{\bar{T}} D_{\bar{T}} \bar{f} \right] - \frac{1}{(T-\bar{T})^2} \left[ U^2 \partial_U D_U f - \bar{U}^2 \partial_{\bar{U}} D_{\bar{U}} \bar{f} \right], \\ \hat{g}_{23}^{-2} &= -(\tilde{S} - \bar{\tilde{S}}) \frac{(T+\bar{T})(U+\bar{U})}{(T-\bar{T})(U-\bar{U})} - \frac{1}{2} [D_U D_T f - D_{\bar{U}} D_{\bar{T}} \bar{f}] \\ &\quad - \frac{1}{(U-\bar{U})^2} [T\bar{U} \partial_T D_T f - \bar{T}U \partial_{\bar{T}} D_{\bar{T}} \bar{f}] - \frac{1}{(T-\bar{T})^2} [\bar{T}U \partial_U D_U f - T\bar{U} \partial_{\bar{U}} D_{\bar{U}} \bar{f}], \end{aligned} \quad (2.9)$$

with the pseudo-invariant dilaton  $\tilde{S}$  [12]

$$\tilde{S} = S + \frac{1}{2} \partial_T \partial_U f. \quad (2.10)$$

### 3. String amplitudes

In this section we determine the one-loop correction to the  $U(1)_L^2$  gauge couplings of the effective action of an N=2 heterotic string compactified on  $K3 \times T^2$  which has been studied in [12] from a field theoretical point of view. Here we want to focus on the derivation via string amplitudes. To this end we calculate the CP even part of two-point one-loop string amplitudes including gauge bosons of the internal Abelian gauge group of the torus  $T^2$  in a background field method. Then we compare the  $\mathcal{O}(k^2)$  piece of the string amplitudes with the effective Lagrangian (2.2) of N=2 supergravity. We also compute three point amplitudes with two gauge bosons and a modulus  $U$  or  $T$  which corresponds to derivatives of the gauge couplings.

The relevant vertex operators in the zero ghost picture for the moduli  $T = T_1 + iT_2 = 2(b + i\sqrt{G})$  and  $U = (G_{12} + i\sqrt{G})/G_{11}$  w.r.t. the background fields  $G_{IJ}$  and  $B_{IJ} = b \epsilon_{IJ}$  are

$$V_{\pm}^{(0)} = \bar{\partial}X^{\pm} \left[ \partial X^{\pm} + i(k \cdot \psi)\Psi^{\pm} \right] e^{ik \cdot X} , \quad (3.1)$$

where  $X^{\pm} = \frac{1}{\sqrt{2}}(X_1 \pm iX_2)$  are the internal bosonic fields,  $\Psi^{\pm}$  their supersymmetric partners and  $\psi^{\mu}$  are spacetime fermions with  $\mu = 0, \dots, 3$ . The vertex operators for the  $U(1)_L^2$  gauge bosons of the internal torus are [14]

$$V_{A\pm}^{(0)} = \rho \epsilon_{\mu} \bar{\partial}X^{\pm} \left[ \partial X^{\mu} + i(k \cdot \psi)\psi^{\mu} \right] e^{ik \cdot X} , \quad (3.2)$$

with spacetime polarization tensor  $\epsilon_{\mu}$  and  $\rho(T) = \sqrt{\frac{U_2}{T_2}}$  and  $\rho(U) = \sqrt{\frac{T_2}{U_2}}$ .

There is an important point regarding the choice of normalization of the vertex operators, which has two, seemingly different, explanations: one based on target-space-duality, i.e. string theory and one coming from N=2 supergravity. The stringy argument: The calculated amplitudes have to have a certain modular weight under  $T$ - and  $U$ -duality as it can be anticipated from (2.9). This is precisely achieved by that choice. The supergravity argument: The specific mixing between the scalars of the vector multiplet and gauge bosons via the covariant derivative involves a coupling which is not the gauge coupling but given by the Kähler metric. If the fields and propagators are correctly normalized the corresponding Feynman diagram contributes only with the gauge coupling.

### 3.1. Two-point string amplitudes

We consider the  $\mathcal{O}(k^2)$  contribution of the two point one-loop string amplitude including two gauge bosons  $A_\mu^+$ . It will produce a term

$$-\frac{i}{4} \frac{\Delta_{(TT)}}{8\pi^2} F_{\mu\nu}^T F^{\mu\nu T} \quad (3.3)$$

in the effective action. We denote the one-loop threshold correction to the internal  $U(1)_T$  gauge coupling by  $\Delta_{(TT)}$ . On the other hand, the gauge couplings (2.6), which refer to the supergravity basis, will turn out to be linear combinations of the couplings of  $U(1)_T$  and  $U(1)_U$ .

We take the gauge boson vertex operators of the two-point function in a constant background field similarly to [15]. Otherwise, the kinematic factor will cause the two point amplitude to vanish. Thus we take  $A_\mu^+(X) = -\frac{1}{2} F_{\mu\nu}^T X^\nu$  with  $F_{\mu\nu}^T = \text{const}$  and the polarization tensor of the gauge boson  $A_\mu^+$  is replaced by  $\epsilon_\mu e^{ik \cdot X} \rightarrow A_\mu^+(X)$ . The vertex operator of the gauge bosons is then  $\tilde{V}_{A_+}^{(0)} = -\frac{1}{2} F_{\mu\nu}^T \rho(T) \bar{\partial} X^+ (X^\nu \partial X^\mu + \psi^\mu \psi^\nu)$ . The general expression for the CP even part of the string amplitude is [16]

$$\mathcal{A}(A_+, A_+) = \sum_{\text{even } s} (-1)^{s_1+s_2} \int_\Gamma d^2\tau Z(\tau, \bar{\tau}, s) \int d^2z_1 \langle \tilde{V}_{A_+}^{(0)}(z, \bar{z}) \tilde{V}_{A_+}^{(0)}(0) \rangle, \quad (3.4)$$

where

$$Z(\tau, \bar{\tau}, s) = \text{Tr}_{s_1} \left[ (-1)^{s_2 F} q^{H-\frac{1}{2}} \bar{q}^{\bar{H}-1} \right] = Z_\psi Z_X Z_{X_0} Z_{\text{int}} \quad (3.5)$$

is the partition function ( $q = e^{2\pi i \tau}$ ,  $\bar{q} = e^{-2\pi i \bar{\tau}}$ ) for even spin structures  $(s_1, s_2) = (1, 0), (0, 0), (0, 1)$  and  $Z_\psi = \frac{\theta_\alpha(0, \tau)}{\eta(\tau)}$  the fermionic partition function where  $\theta_\alpha$  are the Riemann theta functions for  $\alpha = 2, 3, 4$ . The contribution from bosonic zero modes is  $Z_{X_0} = \frac{1}{32\pi^4 \tau_2^2}$  and  $Z_X = \frac{1}{|\eta(\tau)|^4}$  is the bosonic partition function. The fermion number is denoted by  $F$ . The integration region is the fundamental region of the worldsheet torus  $\Gamma = \{\tau : |\tau_1| \leq \frac{1}{2}, |\tau| \geq 1\}$ .

After summing over even spin structures we only get non vanishing contributions if four space-time fermions are contracted because of a theta function identity. Therefore, pure bosonic contractions may be omitted. The two point function gives<sup>2</sup>

$$\langle \tilde{V}_{A_+}^{(0)}(\bar{z}, z) \tilde{V}_{A_+}^{(0)}(0) \rangle = -\frac{1}{2} F_{\mu\nu}^T F^{\mu\nu T} \rho(T)^2 G_F^2 \langle \bar{\partial} X_1^+ \bar{\partial} X_2^+ \rangle, \quad (3.6)$$

---

<sup>2</sup> We introduced the notation  $X_i = X(z_i, \bar{z}_i)$ .

where  $G_F = \frac{\theta_1(0,\tau)\theta_\alpha(z,\tau)}{\theta_\alpha(0,\tau)\theta_1(z,\tau)}$  with  $\alpha = 2, 3, 4$  is the fermionic Green function and the part of  $G_F^2$  depending on spin structures is  $4\pi i \partial_\tau \ln Z_\psi$  which does no longer depend on worldsheet coordinates.  $G_B = -\ln |\chi|^2$  is the bosonic Green function with  $|\chi|^2 = 4\pi^2 e^{-2\pi(\text{Im } z)^2/\text{Im } \tau} \left| \frac{\theta_1(z,\tau)}{\theta_1(0,\tau)} \right|^2$ . We take the following Green functions for the internal bosons

$$\begin{aligned}\langle \bar{\partial} X^\pm \bar{\partial} X^\pm \rangle &= 2\pi^2 (P_R^\pm)^2 \\ \langle \bar{\partial} X^\pm \bar{\partial} X^\mp \rangle &= 2\pi^2 P_R^\pm P_R^\mp - \frac{\pi}{\tau_2} + \bar{\partial}^2 G_B ,\end{aligned}\tag{3.7}$$

with Narain momenta  $P_{R/L}^+ = \bar{P}_{R/L}$  and  $P_{R/L}^- = P_{R/L}$  which are defined as

$$\begin{aligned}P_L &= \frac{1}{\sqrt{2T_2 U_2}} \left( m_1 + m_2 \bar{U} + n_1 \bar{T} + n_2 \bar{T} \bar{U} \right) \\ P_R &= \frac{1}{\sqrt{2T_2 U_2}} \left( m_1 + m_2 \bar{U} + n_1 T + n_2 T \bar{U} \right) .\end{aligned}\tag{3.8}$$

From  $\int d^2 z \partial^2 G_B = 0$  we get  $\int d^2 z \partial^2 \ln \theta_1(z, \tau) = -\pi$ . Using this relation we find the following result

$$\mathcal{A}(A_+, A_+) = -\frac{i}{4} F_{\mu\nu}^T F^{\mu\nu T} \Delta_{(TT)} ,\tag{3.9}$$

with

$$\Delta_{(TT)} = -\frac{U_2}{T_2} \int \frac{d^2 \tau}{\tau_2} \left[ \sum_{P_L, P_R} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} \bar{P}_R^2 \right] \bar{F}_{-2}(\bar{\tau}) ,\tag{3.10}$$

where  $\bar{F}_{-2} = \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}}$  and we define

$$Z_{torus}(\tau, \bar{\tau}) = \sum_{(P_L, P_R)} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} := \sum_{(P_L, P_R)} \hat{Z}_{torus} .\tag{3.11}$$

The string amplitude involving  $\langle \tilde{V}_{A_-}^{(0)} \tilde{V}_{A_-}^{(0)} \rangle$  is easily obtained from (3.9) by exchanging  $T_2$  with  $U_2$  and replacing  $\bar{P}_R$  with its complex conjugate  $P_R$ . Similarly, for the string amplitude  $\langle \tilde{V}_{A_+}^{(0)} \tilde{V}_{A_+}^{(0)} \rangle$  we get the modular invariant result:

$$\mathcal{A}(A_+, A_-) = \frac{i}{4} F_{\mu\nu}^T F^{\mu\nu U} \int \frac{d^2 \tau}{\tau_2} \sum_{(P_L, P_R)} \left( |P_R|^2 - \frac{1}{2\pi\tau_2} \right) \hat{Z}_{torus} \bar{F}_{-2}(\bar{\tau}) .\tag{3.12}$$

This result can be directly compared with the one loop correction to the Kähler potential  $G_{T\bar{T}}^{(1)} = -\frac{i}{2} G_{T\bar{T}}^{(0)} D_T D_{\bar{T}} f + hc$ . which has been derived in [13]. The second part may be identified with the Green-Schwarz (GS)-term  $2G^{(1)} \equiv \Delta_{univ} = \int \frac{d^2 \tau}{\tau_2} \left( -\frac{1}{2\pi\tau_2} \right) Z_{torus} \bar{F}_{-2}$ .

Our results (3.9) and (3.12) refer to the string basis (3.1) and (3.2) and (therefore) involve modular invariant integrands. Since the momenta transform under  $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$ , like  $(ad - bc = 1)$

$$\begin{aligned} (P_L, \bar{P}_R) &\rightarrow \sqrt{\frac{cT+d}{c\bar{T}+d}} (P_L, \bar{P}_R) \quad , \quad T \rightarrow \frac{aT+b}{cT+d} \quad , \quad U \rightarrow U \quad , \\ (P_L, P_R) &\rightarrow \sqrt{\frac{cU+d}{c\bar{U}+d}} (P_L, P_R) \quad , \quad T \rightarrow T \quad , \quad U \rightarrow \frac{aU+b}{cU+d} \quad , \end{aligned} \quad (3.13)$$

we realize that these amplitudes transform with specific weights  $(w_T, w_U) = (2, -2)$ ,  $(w_{\bar{T}}, w_{\bar{U}}) = (0, 0)$  and  $(w_T, w_U) = (0, 0)$ ,  $(w_{\bar{T}}, w_{\bar{U}}) = (0, 0)$ , respectively. They can be directly identified with well-defined integrals  $\mathcal{I}_{2,-2}$  and  $\mathcal{I}_{0,0}$  as will be shown in the next section.

The one-loop correction to the gauge coupling  $g_{22}^{-2}$ , as it has appeared in the last section and which is therefore w.r.t. the supergravity basis, is then obtained by taking a linear combination of string amplitudes (3.9) and (3.12), which corresponds to the correlation function of two gauge boson vertex operators  $A_\mu^T = \frac{1}{2}[A_\mu^+ - \frac{(U-\bar{U})}{(T-\bar{T})}A_\mu^-]$ :

$$\begin{aligned} [g_{22}^{-2}]^{1-loop} &= \frac{1}{4} \left[ \frac{\Delta_{(TT)}}{8\pi^2} - 2 \frac{(U-\bar{U})}{(T-\bar{T})} \frac{\Delta_{(TU)}}{8\pi^2} + \frac{(U-\bar{U})^2}{(T-\bar{T})^2} \frac{\Delta_{(UU)}}{8\pi^2} \right] - \frac{1}{8\pi^2} \frac{U_2}{T_2} G^{(1)} \\ &= -\frac{1}{32\pi^2} \left[ \frac{U-\bar{U}}{T-\bar{T}} \int \frac{d^2\tau}{\tau_2} \sum_{(P_L, P_R)} (\bar{P}_R - P_R)^2 \hat{Z}_{torus} \bar{F}_{-2}(\bar{\tau}) \right]. \end{aligned} \quad (3.14)$$

Notify, that in the above expression the GS-term cancels the one in  $\Delta_{(TU)}$ . After symplectic transformation (2.7) to the gauge coupling  $\hat{g}_{22}^{-2}$  one obtains for its loop-correction:

$$\begin{aligned} [\hat{g}_{22}^{-2}]^{1-loop} &= \frac{\bar{U}^2}{(U-\bar{U})^2} \frac{\Delta_{(TT)}}{8\pi^2} + \frac{U^2}{(T-\bar{T})^2} \frac{\Delta_{(UU)}}{8\pi^2} + 2 \frac{|U|^2}{(U-\bar{U})(T-\bar{T})} \frac{\Delta_{(TU)}}{8\pi^2} \\ &\quad + \frac{|U|^2}{(T-\bar{T})(U-\bar{U})} \frac{G^{(1)}}{2\pi^2} \\ &= -\frac{1}{8\pi^2} \left[ \frac{1}{(T-\bar{T})(U-\bar{U})} \int \frac{d^2\tau}{\tau_2} \sum_{(P_L, P_R)} (\bar{U}\bar{P}_R + UP_R)^2 \hat{Z}_{torus} \bar{F}_{-2}(\bar{\tau}) \right]. \end{aligned} \quad (3.15)$$

This amplitude can be directly derived from a two point amplitude with vertex operators corresponding to  $\hat{A}_\mu^T = \frac{\bar{U}}{(U-\bar{U})}A_\mu^+ - \frac{U}{(T-\bar{T})}A_\mu^-$ . Using the results from the next section we may directly cast (3.14) and (3.15) into the forms (2.6) and (2.9), dictated by supergravity.

In the corrections (3.14) and (3.15), there appears the non-modular invariant GS-term  $G^{(1)}$ . On the other hand, the tree-level dilaton field gets modified at one-loop by

the same amount with an opposite sign [17]. It is the one-loop dilaton  $S$  which appears in (2.9):

$$g_{\text{string}}^{-2} = \frac{S - \overline{S}}{2} - \frac{1}{16\pi^2} G^{(1)} . \quad (3.16)$$

Thus, altogether, the physical coupling stays modular invariant. See also [18] for a more complete discussion.

### 3.2. Three-point amplitudes

Now we consider three point amplitudes which involve two internal gauge bosons and one modulus. First we want to investigate the amplitude including two gauge bosons  $A_T$  and one  $T$  modulus which is related to  $\partial_T \Delta_{(TT)} F_{\mu\nu} F^{\mu\nu} T$  in the effective string action, where  $\partial_T \Delta_{(TT)}$  denotes the derivative with respect to  $T$  of the one loop correction to the  $U(1)_T$  gauge coupling. The correlation function gives the following contractions:

$$\begin{aligned} \langle V_+^{(0)}(z_1) V_{A+}^{(0)}(z_2) V_{A+}^{(0)}(z_3) \rangle &= \mathcal{K} G_F^2 \prod_{i < j} |\chi_{ij}|^{2k_i \cdot k_j} \frac{U_2}{iT_2^2} \\ &\quad \left( \langle \bar{\partial} X_1^+ \partial X_1^+ \rangle \langle \bar{\partial} X_2^+ \bar{\partial} X_3^+ \rangle + \langle \bar{\partial} X_1^+ \bar{\partial} X_2^+ \rangle \langle \partial X_1^+ \bar{\partial} X_3^+ \rangle \right. \\ &\quad \left. + \langle \bar{\partial} X_1^+ \bar{\partial} X_3^+ \rangle \langle \partial X_1^+ \bar{\partial} X_2^+ \rangle \right), \end{aligned} \quad (3.17)$$

where  $\mathcal{K} = \left( (k_2 k_3)(\epsilon_2 \epsilon_3) - (k_2 \epsilon_3)(k_3 \epsilon_2) \right)$  is the kinematic factor.

Before doing the worldsheet integrals we want to make some comments on possible additional non trivial contributions to the  $\mathcal{O}(k^2)$  part of the amplitude. We may get contributions from the delta function which might appear in the correlation function  $\langle \bar{\partial} X^\pm \partial X^\mp \rangle$ . If we consider the region  $|z_{ij}| < \epsilon$  then  $|\chi_{ij}| \simeq |z_{ij}|$  and thus the delta function can be omitted because  $\int d^2 z_i \delta^{(2)}(z_{ij}) |z_{ij}|^{2k_i \cdot k_j} f(z_{ik}) = 0$  where  $f$  is some function. But if  $|z_{ij}| > \epsilon$  and  $|k_i \cdot k_j G_{ij}^B| \ll 1$  then one can expand  $|\chi_{ij}|^{2k_i \cdot k_j} = 1 - k_i \cdot k_j G_{ij}^B + \dots$  and in this case one indeed gets contributions from the delta function for the lowest term of the expansion [19]. On the other hand, if the correlation functions can be approximated such that the worldsheet integral gives  $\int_{|z_{il}| < \epsilon} d^2 z_{il} \frac{|z_{il}|^{2k_i \cdot k_l}}{|z_{il}|^2} \sim \frac{\pi}{k_i \cdot k_l}$  one may e.g. produce a  $\mathcal{O}(k^2)$  contribution from terms of the order  $\mathcal{O}(k^4)$ . These contributions are important when one has to collect all possible terms of a particular order [20]. But in the case considered here, pinched off integrals only give  $\int_{|z_{il}| < \epsilon} d^2 z_{il} \frac{|z_{il}|^{2k_i \cdot k_l}}{|z_{il}|^4} \simeq \frac{\pi}{k_i \cdot k_l - 1}$  and thus do not contribute to the  $\mathcal{O}(k^2)$  piece of the amplitude.

In the following we will restrict ourselves to the region  $|z_{ij}| < \epsilon$ . Taking into account the arguments mentioned above it remains to perform the worldsheet integral of (3.17). We end up with:

$$\mathcal{A}(T, A_+, A_+)|_{\mathcal{O}(k^2)} = -\mathcal{K} \frac{\pi^2 U_2}{2 T_2^2} \int d^2\tau \left\{ \sum_{P_L, P_R} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} \bar{P}_R^3 P_L \right\} \bar{F}_{-2}(\bar{\tau}) \quad (3.18)$$

This term can be identified with the third derivative of the prepotential<sup>3</sup>  $f_{TTT}$  which has been derived in [13] by taking particular derivatives on the integral coming from a CP odd string amplitude of the one loop correction to the Kähler potential  $G_{T\bar{T}}^{(1)}$ . Thus one finds

$$\mathcal{A}(T, A_T, A_T)|_{\mathcal{O}(k^2)} = -4i\mathcal{K}\pi^3 f_{TTT} . \quad (3.19)$$

We will have to say more about this result in section 4. We realize that this expression transforms covariantly under  $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$  with weights  $(w_T = 4, w_{\bar{T}} = 0)$  and  $(w_U = -2, w_{\bar{U}} = 0)$ , respectively.

Besides we calculate the three point amplitudes  $\langle V_+^{(0)} V_{A_+}^{(0)} V_{A_-}^{(0)} \rangle$  and  $\langle V_+^{(0)} V_{A_-}^{(0)} V_{A_-}^{(0)} \rangle$  with the result

$$\begin{aligned} \mathcal{A}(T A_+ A_-) &= -\frac{\mathcal{K}\pi^2}{2 T_2} \int d^2\tau \left\{ \sum_{P_L, P_R} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} \left[ \bar{P}_R |P_R|^2 P_L - \frac{1}{\pi\tau_2} \bar{P}_R P_L \right] \right\} \bar{F}_{-2}(\bar{\tau}) \\ \mathcal{A}(T A_- A_-) &= -\frac{\mathcal{K}\pi^2}{2 U_2} \int d^2\tau \left\{ \sum_{P_L, P_R} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} \left[ P_R |P_R|^2 P_L - \frac{1}{\pi\tau_2} P_R P_L \right] \right\} \bar{F}_{-2}(\bar{\tau}) . \end{aligned} \quad (3.20)$$

These amplitudes can be casted into the convenient form:

$$\begin{aligned} \mathcal{A}(T A_+ A_-) &= -4i\mathcal{K}\pi^3 \left[ f_{TTU} + \frac{1}{4\pi^2} G_T^{(1)} \right] \\ \mathcal{A}(T A_- A_-) &= -4i\mathcal{K}\pi^3 \left[ f_{TUU} + \frac{1}{4\pi^2} G_U^{(1)} \right] . \end{aligned} \quad (3.21)$$

The linear combination which corresponds to the three point amplitude with the  $T$  modulus and two  $A_\mu^T$  gauge bosons is

$$\begin{aligned} \mathcal{A}(T A^T A^T) &= \frac{1}{4} \left[ \mathcal{A}(T A_+ A_+) - 2 \frac{(U - \bar{U})}{(T - \bar{T})} \mathcal{A}(T A_+ A_-) + \frac{(U - \bar{U})^2}{(T - \bar{T})^2} \mathcal{A}(T A_- A_-) \right] \\ &= -\frac{\mathcal{K}\pi^2 U_2}{8 T_2^2} \int d^2\tau \left\{ \sum_{P_L, P_R} q^{\frac{1}{2}|P_L|^2} \bar{q}^{\frac{1}{2}|P_R|^2} P_L \bar{P}_R (\bar{P}_R - P_R)^2 \right. \\ &\quad \left. + \frac{2}{\pi\tau_2} P_L \bar{P}_R - \frac{1}{\pi\tau_2} P_L P_R \right\} \bar{F}_{-2}(\bar{\tau}) . \end{aligned} \quad (3.22)$$

---

<sup>3</sup> The relevant relations between the prepotential and  $\tau$ -integrals may be found in the next section.

The string amplitude is not a 1PI diagram but also contains other exchange diagrams and therefore splits into field theoretical amplitudes containing one loop corrections to the gauge coupling.

$$\mathcal{A}(TA^T A^T) = \frac{\mathcal{K}\pi U_2}{4iT_2} \left[ 16\pi^2 \partial_T g_{TU}^{-2} - \frac{1}{4iU_2} \Delta_{(TT)} + \frac{U_2}{4iT_2^2} \Delta_{(UU)} \right] . \quad (3.23)$$

#### 4. Prepotential and world-sheet torus integrations

In this section we want to find relations of the one-loop prepotential  $f$  and/or derivatives of it to world-sheet  $\tau$ -integrals as they appear in the previous section. The one-loop correction<sup>4</sup> to the heterotic prepotential (2.1) can be written in the chamber  $T_2 > U_2$  [10]

$$f(T, U) = \frac{i}{6\pi} U^3 + \frac{2}{(2\pi i)^4} \sum_{(k,l)>0} c_1(kl) \mathcal{L}i_3 \left[ e^{2\pi i(kT+lU)} \right] + \frac{1}{(2\pi)^4} c_1(0) \zeta(3) , \quad (4.1)$$

where the numbers  $c_1(n)$  are related to the (new) supersymmetric index

$$\mathcal{Z}(\tau, \bar{\tau}) = \bar{\eta}^{-2} \text{Tr}_R \left[ F(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right] , \quad (4.2)$$

which for heterotic compactifications on  $K3 \times T^2$  with the choice of  $SU(2)$  instanton numbers  $(12, 12)$ ,  $(11, 12)$ ,  $(10, 14)$  and  $(24, 0)$  becomes [10]

$$\begin{aligned} \mathcal{Z}(\tau, \bar{\tau}) &= 2i Z_{torus}(\tau, \bar{\tau}) \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} , \\ \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} &= \sum_{n \geq -1} c_1(n) \bar{q}^n . \end{aligned} \quad (4.3)$$

The mentioned models lead to the gauge group  $E_7 \times E_7$  and  $E_7 \times E_8$ , respectively. In the first three cases the gauge group may be completely Higgsed away. At the perturbative level these models are equivalent. A fact, which also becomes clear from the unique expression for the supersymmetric index (4.3) which enters all kinds of perturbative string calculations (cf. e.g. the previous section). These three models (after Higgsing completely) are dual to type IIA Calabi–Yau compactifications, which are elliptic fibrations over the Hirzebruch surfaces  $\mathbf{F}_0, \mathbf{F}_1, \mathbf{F}_2$ . Then the holomorphic part (to be identified with the Wilsonian coupling) of the three-point functions (3.18) and (3.20) [in particular (3.18)]

---

<sup>4</sup> Compared to the previous sections we now change  $f \rightarrow if$ .

and (3.20)] is related to the Yukawa couplings  $f_{TTT}, f_{TTU}$  and  $f_{TUU}$  of the Calabi–Yau manifold, respectively [8][9]. Moreover using mirror symmetry these couplings are given by the classical intersection numbers of the typeIIB theory. Supersymmetric indices (4.2), valid for the other bases  $\mathbf{F}_k$  are the subject of [21]. They allow for more general instanton embeddings and one ends up with larger terminal gauge groups after Higgsing.

We introduce the polylogarithms ( $a \geq 1$ ):

$$\mathcal{L}i_a(x) = \sum_{p>0} \frac{x^p}{p^a} . \quad (4.4)$$

The integrals we should look for involve Narain momenta from ‘charge’ insertions or zero–mode contributions. In general they show up in string amplitudes involving  $U(1)$ –charges w.r.t. the internal bosonic fields or after contractions of bosonic internal fields (belonging to the  $T^2$ ). I.e. we consider  $(\alpha, \beta, \gamma, \delta \geq 0)$

$$\mathcal{I}_{w_T, w_U} := (T - \bar{T})^m (U - \bar{U})^n \int \frac{d^2\tau}{\tau_2^k} \sum_{(P_L, P_R)} P_L^\alpha P_R^\beta \bar{P}_L^\gamma \bar{P}_R^\delta \hat{Z}_{torus}(\tau, \bar{\tau}) \bar{F}_l(\bar{\tau}) . \quad (4.5)$$

We want this expression to have modular weights  $(w_T, w_U)$  and weights  $(w_{\bar{T}}, w_{\bar{U}}) = (0, 0)$  under  $T, U$ –duality. This imposes the conditions [cf. (3.13)]:

$$\begin{aligned} m &= -\frac{w_T}{2} \\ n &= -\frac{w_U}{2} \\ \gamma &= \alpha - \frac{1}{2}(w_T + w_U) \\ \delta &= \beta + \frac{1}{2}(w_T - w_U) . \end{aligned} \quad (4.6)$$

There is also a relation for  $k$  and  $l$  which follows from modular invariance of the integrand, which can be easily deduced after a Poisson resummation on the momenta  $m_i$ . Since the integrals (4.5) will be constructed such that they transform with a certain weight under  $SL(2, \mathbf{Z})_T \times SL(2, \mathbf{Z})_U$  we expect that  $\mathcal{I}_{w_T, w_U}$  can be written in terms of modular covariant derivatives of  $f$  rather than usual derivatives. The prepotential  $f(T, U)$  has weights  $(w_T, w_U) = (-2, -2)$ . Acting with the covariant derivative (cf. also section 2)

$$D_T = \partial_T - \frac{2}{T - \bar{T}} \quad (4.7)$$

increases its weight  $w_T$  by 2. In general with the derivative

$$D_T^n = \partial_T - \frac{2n}{T - \bar{T}} \quad (4.8)$$

one changes the weight from  $-2n$  to  $-2n + 2$ . This derivative is also covariant w.r.t. the Kähler connection  $a_\mu \sim [\partial_i K(\Phi, \bar{\Phi}) D_\mu \phi^i - \partial_{\bar{i}} K(\Phi, \bar{\Phi}) D_\mu \bar{\phi}^{\bar{i}}]$ , which means from the point of view of amplitudes that one-particle reducible diagrams with massless states running in the loop are subtracted to end up with the 1PI effective action.

In subsection 4.1 we consider cases involving only two momenta, i.e.  $\alpha + \beta + \gamma + \delta = 2$ . In subsection 4.2. some cases of more than two momenta insertions.

#### 4.1. Two momenta insertions

##### 4.1.1. $f_{TT}$

Let us consider the integral

$$\mathcal{I}_{2,-2} := \frac{(U - \bar{U})}{(T - \bar{T})} \int \frac{d^2 \tau}{\tau_2} \sum_{(P_L, P_R)} \bar{P}_R^2 \hat{Z}_{torus}(\tau, \bar{\tau}) \bar{F}_{-2}(\bar{\tau}) , \quad (4.9)$$

which after a Poisson resummation on  $m_i$  (cf. Appendix A) becomes

$$\mathcal{I}_{2,-2} = \frac{(U - \bar{U})}{(T - \bar{T})} T_2^2 \int \frac{d^2 \tau}{\tau_2^4} \sum_{\substack{n_1, n_2 \\ l_1, l_2}} \bar{Q}_R^2 e^{-2\pi i \bar{T} \det A} e^{\frac{-\pi T_2}{\tau_2 U_2} |n_1 \tau + n_2 U \tau - U l_1 + l_2|^2} \bar{F}_{-2}(\bar{\tau}) . \quad (4.10)$$

with  $A = \begin{pmatrix} n_1 & -l_2 \\ n_2 & l_1 \end{pmatrix} \in M(2, \mathbf{Z})$  and we introduce:

$$\begin{aligned} Q_R &= \frac{1}{\sqrt{2T_2 U_2}} [(n_2 \bar{U} + n_1) \tau - \bar{U} l_1 + l_2] \\ \bar{Q}_R &= \frac{1}{\sqrt{2T_2 U_2}} [(n_2 U + n_1) \tau - U l_1 + l_2] \\ Q_L &= \frac{1}{\sqrt{2T_2 U_2}} [(n_2 \bar{U} + n_1) \bar{\tau} - \bar{U} l_1 + l_2] \\ \bar{Q}_L &= \frac{1}{\sqrt{2T_2 U_2}} [(n_2 U + n_1) \bar{\tau} - U l_1 + l_2] . \end{aligned} \quad (4.11)$$

In that form (4.10) one easily checks modular invariance, i.e. one deduces the only possible choice for  $k$  and  $l$  in (4.5). For the anti-holomorphic function we choose

$$\bar{F}_{-2}(\bar{\tau}) = \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} \quad (4.12)$$

as it arises in physical amplitudes (cf. e.g. section 3). Modular invariance also enables us to use the orbit decomposition used in [16], i.e. decomposing the set of all matrices  $A$  into orbits of  $SL(2, \mathbf{Z})$ :

$$\begin{aligned} I_0 : A &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \\ I_1 : A &= \pm \begin{pmatrix} k & j \\ 0 & p \end{pmatrix} , \quad 0 \leq j < k , \quad p \neq 0 , \\ I_2 : A &= \begin{pmatrix} 0 & j \\ 0 & p \end{pmatrix} , \quad (j, p) \neq (0, 0) . \end{aligned} \quad (4.13)$$

Clearly,  $I_0$  does not give any contribution. The remaining  $\tau$ -integrals  $I_1$  and  $I_2$  are presented in appendix B and we evaluate for (4.9):

$$\mathcal{I}_{2,-2} = 8\pi^2 \partial_T \left( \partial_T - \frac{2}{T - \bar{T}} \right) f + 8\pi^2 \frac{(U - \bar{U})^2}{(T - \bar{T})^2} \partial_{\bar{U}} \left( \partial_{\bar{U}} + \frac{2}{U - \bar{U}} \right) \bar{f} . \quad (4.14)$$

It is quite remarkable, how e.g. the cubic term of the prepotential (4.1), in the combination of (4.14), gives the last term of (B.19).

#### 4.1.2 $f_{UU}$

Similary, an expression with modular weights  $(w_T, w_U) = (-2, 2)$  can be found:

$$\mathcal{I}_{-2,2} := \frac{(T - \bar{T})}{(U - \bar{U})} \int \frac{d^2 \tau}{\tau_2} \sum_{(P_L, P_R)} P_R^2 \hat{Z}_{torus}(\tau, \bar{\tau}) \bar{F}_{-2}(\bar{\tau}) , \quad (4.15)$$

which after a Poisson resummation on  $m_i$  (cf. Appendix) becomes

$$\mathcal{I}_{-2,2} = \frac{(T - \bar{T})}{(U - \bar{U})} T_2^2 \int \frac{d^2 \tau}{\tau_2^4} \sum_{\substack{n_1, n_2 \\ l_1, l_2}} Q_R^2 e^{-2\pi i \bar{T} \det A} e^{\frac{-\pi T_2}{\tau_2 U_2} |n_1 \tau + n_2 U \tau - U l_1 + l_2|^2} \bar{F}_{-2}(\bar{\tau}) . \quad (4.16)$$

After the integration we end up with:

$$\mathcal{I}_{-2,2} = 8\pi^2 \partial_U \left( \partial_U - \frac{2}{U - \bar{U}} \right) f + 8\pi^2 \frac{(T - \bar{T})^2}{(U - \bar{U})^2} \partial_{\bar{T}} \left( \partial_{\bar{T}} + \frac{2}{T - \bar{T}} \right) \bar{f} . \quad (4.17)$$

Alternatively, with mirror symmetry  $T \leftrightarrow U$ , which induces the action  $P_L \leftrightarrow P_L, P_R \leftrightarrow \bar{P}_R$  on the Narain momenta one may obtain (4.17) from (4.9).

#### 4.1.3 $f_{TU}$

There are several ways to construct from (4.5)  $\tau$ -integrals of weights  $w_T = 0, w_U = 0$  which involve at most two Narain momenta insertions. Let us take

$$\mathcal{I}_{0,0}^a := \int \frac{d^2\tau}{\tau_2} \sum_{(P_L, P_R)} \hat{Z}_{torus}(\tau, \bar{\tau}) \bar{F}_0(\tau, \bar{\tau}) \quad , \quad (4.18)$$

with  $(\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2})$

$$\bar{F}_0(\tau, \bar{\tau}) = \lambda_1 \frac{\hat{E}_2 \bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} + \lambda_2 \frac{\bar{E}_6^2}{\bar{\eta}^{24}} + \lambda_3 \frac{\bar{E}_4^3}{\bar{\eta}^{24}} \quad (4.19)$$

and we choose  $-264\lambda_1 - 984\lambda_2 + 744\lambda_3 = 0$  to avoid holomorphic anomalies arising from triangle graphs involving two gauge fields and the Kähler- or sigma model connection as external legs with massless states running in the loop. In other words, we want to discard non-harmonic  $\ln(T - \bar{T})(U - \bar{U})$  terms. Later we will see that this combination is precisely related to the ‘physical’ choice (4.12) for  $(\lambda_1, \lambda_2, \lambda_3) = (1/6, 1/3, 1/2)$ . In fact:

$$\frac{1}{2\pi i} \frac{\partial}{\partial \bar{\tau}} \bar{F}_{-2}(\bar{\tau}) = \frac{1}{6} \frac{\bar{E}_2 \bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} + \frac{1}{3} \frac{\bar{E}_6^2}{\bar{\eta}^{24}} + \frac{1}{2} \frac{\bar{E}_4^3}{\bar{\eta}^{24}} \quad . \quad (4.20)$$

The expression  $\mathcal{I}_{0,0}^a$  gives a representation for a weight zero automorphic form. However, non-harmonic, because of  $\hat{E}_2$  in (4.20). Therefore the theorem of Borchers [22] does not apply. Using results of [10] it easily can be evaluated<sup>5</sup>:

$$\mathcal{I}_{0,0}^a = 16\pi^2 \text{Re}(f_{TU}) + 2G^{(1)} \quad . \quad (4.21)$$

Using the explicit form of the N=2 version of the GS-term [10]

$$G^{(1)} = \frac{32\pi^2}{(T - \bar{T})(U - \bar{U})} \text{Re} \left\{ f - \frac{1}{2}(T - \bar{T})\partial_T f - \frac{1}{2}(U - \bar{U})\partial_U f \right\} \quad , \quad (4.22)$$

we arrive at

---

<sup>5</sup> In [10] the N=2 Green-Schwarz term  $G^{(1)}$  is denoted by  $\Delta_{\text{univ}}$ .

$$\text{Re} \{D_T D_U f\} = \frac{1}{16\pi^2} \int \frac{d^2\tau}{\tau_2} \sum_{(P_L, P_R)} \hat{Z}_{torus}(\tau, \bar{\tau}) \left[ \frac{1}{6} \frac{\hat{\bar{E}}_2 \bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} + \frac{1}{3} \frac{\bar{E}_6^2}{\bar{\eta}^{24}} + \frac{1}{2} \frac{\bar{E}_4^3}{\bar{\eta}^{24}} \right] . \quad (4.23)$$

In addition, we want to investigate the integral

$$\mathcal{I}_{0,0}^b := \int \frac{d^2\tau}{\tau_2} \sum_{(P_L, P_R)} \left( |P_R|^2 - \frac{1}{2\pi\tau_2} \right) \hat{Z}_{torus}(\tau, \bar{\tau}) \bar{F}_{-2}(\bar{\tau}) , \quad (4.24)$$

The additional GS-like term is needed to guarantee modular invariance. That can be seen after performing the Poisson resummation, which yields:

$$\mathcal{I}_{0,0}^b = -T_2^2 \int \frac{d^2\tau}{\tau_2^4} \sum_{\substack{n_1, n_2 \\ l_1, l_2}} Q_R \bar{Q}_R e^{-2\pi i \bar{T} \det A} e^{\frac{-\pi T_2}{\tau_2 U_2} |n_1 \tau + n_2 U \tau - U l_1 + l_2|^2} \bar{F}_{-2}(\bar{\tau}) . \quad (4.25)$$

Again, this integral can be determined using formulas of the appendix B with the result:

$$\mathcal{I}_{0,0}^a = \mathcal{I}_{0,0}^b . \quad (4.26)$$

Let us now come to the identity (4.20), which is the link to (4.26). Rewriting  $\mathcal{I}_{0,0}^b$  as

$$\mathcal{I}_{0,0}^b = \frac{i}{\pi} \int \frac{d^2\tau}{\tau_2^2} \partial_{\bar{\tau}}(\tau_2 Z_{torus}) \bar{F}_{-2}(\bar{\tau}) \quad (4.27)$$

and using (4.20) we may also deduce (4.26) after partial integration.

#### 4.2. More than two momenta insertions

Let us give some representative examples.

##### 4.2.1. $f_{TTT}$

We want to consider the integral

$$\mathcal{I}_{4,-2} := \frac{(U - \bar{U})}{(T - \bar{T})^2} \int d^2\tau \sum_{(P_L, P_R)} P_L \bar{P}_R^3 \hat{Z}_{torus}(\tau, \bar{\tau}) \bar{F}_{-2}(\bar{\tau}) . \quad (4.28)$$

With the identity

$$P_L \bar{P}_R^3 \hat{Z}_{torus} = \frac{T - \bar{T}}{2\pi\tau_2} \bar{P}_R^2 \partial_T \hat{Z}_{torus} \quad (4.29)$$

we are able to ‘transform’ (4.28) into a two-momentum integral of the kind we have discussed before. In particular, this identity tells us

$$2\pi\mathcal{I}_{4,-2} = \left( \partial_T + \frac{2}{T - \bar{T}} \right) \mathcal{I}_{2,-2} . \quad (4.30)$$

Using (4.14) we obtain after some straightforward algebraic manipulations:

$$\mathcal{I}_{4,-2} = 4\pi f_{TTT} . \quad (4.31)$$

This identity was already derived in [13], however in a quite different manner. Moreover, we also may directly integrate (4.28) as we have done so in the last subsections. After a Poisson resummation [cf. (A.1) and (A.5)] the integral (4.28) becomes :

$$\begin{aligned} \mathcal{I}_{4,-2} = & -\frac{(U - \bar{U})}{(T - \bar{T})^2} T_2^4 \int \frac{d^2\tau}{\tau_2^5} \sum_{\substack{n_1, n_2 \\ l_1, l_2}} Q_L \bar{Q}_R^3 e^{-2\pi i \bar{T} \det A} e^{\frac{-\pi T_2}{\tau_2 U_2} |n_1 \tau + n_2 U \tau - U l_1 + l_2|^2} \bar{F}_{-2}(\bar{\tau}) \\ & + \frac{3}{2\pi} \frac{(U - \bar{U})}{(T - \bar{T})^2} T_2^2 \int \frac{d^2\tau}{\tau_2^4} \sum_{\substack{n_1, n_2 \\ l_1, l_2}} \bar{Q}_R^2 e^{-2\pi i \bar{T} \det A} e^{\frac{-\pi T_2}{\tau_2 U_2} |n_1 \tau + n_2 U \tau - U l_1 + l_2|^2} \bar{F}_{-2}(\bar{\tau}) . \end{aligned} \quad (4.32)$$

The second integral is of the kind (4.16). In fact, using the results of appendix B, we have explicitly evaluated the integrals (4.32) and checked (4.31).

#### 4.2.2. $f_{TTU}$

A covariant expression  $\mathcal{I}_{2,0}$  containing  $f_{TTU}$  may be found by considering  $\partial_T \mathcal{I}_{0,0}$ :

$$\mathcal{I}_{2,0} := \partial_T \mathcal{I}_{0,0}^b = \frac{2\pi}{(T - \bar{T})} \int d^2\tau \sum_{(P_L, P_R)} \left( P_L P_R \bar{P}_R^2 - \frac{P_L \bar{P}_R}{\pi \tau_2} \right) \hat{Z}_{torus}(\tau, \bar{\tau}) \bar{F}_{-2}(\bar{\tau}) . \quad (4.33)$$

Whereas in (4.33) each term alone already has weights  $(w_T, w_U) = (2, 0)$  and  $(w_{\bar{T}}, w_{\bar{U}}) = (0, 0)$ , only their combination gives rise to a modular invariant integrand. This may be seen after doing the Poisson transformation:

$$\begin{aligned} \mathcal{I}_{2,0} = & \frac{2\pi}{(T - \bar{T})} T_2^4 \int \frac{d^2\tau}{\tau_2^5} \sum_{\substack{n_1, n_2 \\ l_1, l_2}} Q_L Q_R \bar{Q}_R^2 e^{-2\pi i \bar{T} \det A} e^{\frac{-\pi T_2}{\tau_2 U_2} |n_1 \tau + n_2 U \tau - U l_1 + l_2|^2} \bar{F}_{-2}(\bar{\tau}) \\ & - \frac{2}{(T - \bar{T})} T_2^2 \int \frac{d^2\tau}{\tau_2^4} \sum_{\substack{n_1, n_2 \\ l_1, l_2}} Q_R \bar{Q}_R e^{-2\pi i \bar{T} \det A} e^{\frac{-\pi T_2}{\tau_2 U_2} |n_1 \tau + n_2 U \tau - U l_1 + l_2|^2} \bar{F}_{-2}(\bar{\tau}) . \end{aligned} \quad (4.34)$$

Again, for the integration we use the formulas of appendix B to arrive at:

$$\begin{aligned}\mathcal{I}_{2,0} &= 8\pi^2 \partial_T \left( \partial_T - \frac{2}{T - \bar{T}} \right) \left( \partial_U - \frac{2}{U - \bar{U}} \right) f - \frac{16\pi^2}{(T - \bar{T})^2} \left( \partial_{\bar{U}} + \frac{2}{U - \bar{U}} \right) \bar{f} \\ &= 8\pi^2 f_{TTU} + 2G_T^{(1)}.\end{aligned}\quad (4.35)$$

The second term in the integrand of (4.33) might be identified with  $4G_T^{(1)}$ , although a splitting of both terms does not make sense because of modular invariance. Besides, only the combination  $8\pi^2 f_{TTU} + 2G_T^{(1)}$  can be written covariant w.r.t. (4.8).

#### 4.2.3. $f_{TUU}$

Finally, for the integral

$$\mathcal{I}_{0,2} := \frac{2\pi}{(U - \bar{U})} \int d^2\tau \sum_{(P_L, P_R)} \left( P_L \bar{P}_R P_R^2 - \frac{P_L P_R}{\pi\tau_2} \right) \hat{Z}_{torus}(\tau, \bar{\tau}) \bar{F}_{-2}(\bar{\tau}), \quad (4.36)$$

we borrow the results of section (4.2.2) and use mirror symmetry  $T \leftrightarrow U$  to obtain:

$$\begin{aligned}\mathcal{I}_{0,2} &= 8\pi^2 \partial_U \left( \partial_U - \frac{2}{U - \bar{U}} \right) \left( \partial_T - \frac{2}{T - \bar{T}} \right) f - \frac{16\pi^2}{(U - \bar{U})^2} \left( \partial_{\bar{T}} + \frac{2}{T - \bar{T}} \right) \bar{f} \\ &= 8\pi^2 f_{TUU} + 2G_U^{(1)}.\end{aligned}\quad (4.37)$$

## 5. Six dimensional origin of the gauge couplings

Let us consider the amplitudes discussed in the section 3 from a more general point of view. The gauge kinetic terms (2.2) are deduced from the Einstein term in six dimensions upon dimensional reduction. In the Einstein frame the latter does not receive any loop corrections neither in  $d = 6$  nor in  $d = 4$ . The relevant object to consider is a two graviton amplitude ( $i = 1, \dots, 6$ )

$$\langle : \epsilon_{ij} \bar{\partial} X_1^i \left[ \partial X_1^j + i(k_1 \cdot \psi_1) \psi_1^j \right] e^{ik_1 \cdot X_1} :: \epsilon_{kl} \bar{\partial} X_2^k \left[ \partial X_2^l + i(k_2 \cdot \psi_2) \psi_2^l \right] e^{ik_2 \cdot X_2} : \rangle, \quad (5.1)$$

which may contain both  $\mathcal{R}$  and  $\mathcal{R}_{ikjl} \mathcal{R}^{ikjl}$  corrections. Here  $\epsilon_{ij}$  is the gravitational polarization tensor in six dimensions. The amplitude is determined by expanding the elliptic genus  $\mathcal{A}$  of  $K3$  w.r.t.  $\mathcal{R}^2$ . In general, in  $N=1, d = 6$  heterotic string theories only the 4-form part of the elliptic genus gives rise to modular invariant one-loop corrections [23][24].

For the choice of instanton numbers  $(n_1, n_2) = (24, 0)$  w.r.t. an  $SU(2)$  gauge bundle which leads to the gauge group  $E_7 \times E'_8$  this expansion is given by [23][24][10]

$$\mathcal{A}(\tau, \mathcal{F}, \mathcal{R})|_{4-form} \sim \left[ (\mathcal{R}^2 - \mathcal{F}^2) \frac{\hat{\bar{E}}_2 \bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} + \frac{\bar{E}_6^2}{\bar{\eta}^{24}} \mathcal{F}_{E_7}^2 + \frac{\bar{E}_4^3}{\bar{\eta}^{24}} \mathcal{F}_{E'_8}^2 \right], \quad (5.2)$$

To saturate the fermionic zero modes one has to contract the four fermions which is already of the order  $\mathcal{O}(k^2)$ . Since the worldsheet integral of the bosonic contraction  $\langle \bar{\partial} X_1^i \bar{\partial} X_2^k \rangle$  gives a zero result and thus the  $\mathcal{O}(k^2)$  term of the effective action vanishes, the next to leading order arises from contractions of  $\bar{\partial} X^i$  with the exponential  $e^{ik \cdot X}$  which contributes another  $\mathcal{O}(k^2)$  term to the amplitude. For the  $\mathcal{O}(k^4)$  contribution we thus obtain [24]:

$$\Delta_{\mathcal{R}^2}^{6d, N=1} \sim \int \frac{d^2 \tau}{\tau_2^2} \left( \bar{E}_2 - \frac{3}{\pi \tau_2} \right) \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} = -8\pi. \quad (5.3)$$

The  $\hat{\bar{E}}_2$ -piece arises from the worldsheet integral over the contractions of  $6d$  space-time bosons:

$$\int d^2 z_{12} \langle \bar{\partial} \bar{X}_1^i X_2^m \rangle \langle \bar{\partial} \bar{X}_2^k X_1^n \rangle = -\eta^{im} \eta^{kn} \int \left[ \bar{\partial} G_B(z_{12}) \right]^2 = \eta^{im} \eta^{kn} \frac{\pi^2 \tau_2}{8} \left( \bar{E}_2 - \frac{3}{\pi \tau_2} \right). \quad (5.4)$$

The appearance of  $\hat{\bar{E}}_2$  in (5.3) may be also understood as the gravitational charge  $Q_{\text{grav}}^2 = -2\partial_\tau \ln \eta(\tau) = 1/6\pi i E_2(\tau)$  [25]. After the torus compactification we obtain in  $d = 4$  [24][25][10]

$$\Delta_{\mathcal{R}^2}^{4d, N=2} \sim \int \frac{d^2 \tau}{\tau_2} Z_{\text{torus}} \left( \bar{E}_2 - \frac{3}{\pi \tau_2} \right) \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}}. \quad (5.5)$$

There is however a subtlety w.r.t. to the indices  $i, j, k, l$  in deducing the field theoretic kinematic content of (5.1) for the four dimensional action: Taking all  $i, j, k, l$  as  $d = 4$  space-time indices  $\mu, \nu$  gives (5.3) for the  $\mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma}$  correction in  $4d$ , i.e. (5.5) after taking into account the zero modes of the internal torus. However when we want to deduce the gauge kinetic term (3.3), we keep both  $i$  and  $k$  as internal indices + which has been defined in section 3 and extract the  $\mathcal{O}(k^2)$ -piece of (5.1) [cf. (3.7)]:

$$\int d^2 z_{12} \langle \bar{\partial} X_1^+ \bar{\partial} X_2^+ \rangle = 2\pi^2 \tau_2 \bar{P}_R^2. \quad (5.6)$$

This way we end up at

$$\Delta_{(TT)} \sim \int \frac{d^2 \tau}{\tau_2} \sum_{(P_L, P_R)} \hat{Z}_{\text{torus}} \bar{P}_R^2 \frac{\bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}}. \quad (5.7)$$

after taking into account all kinematic possibilities, in agreement with (3.9). Of course, the  $\bar{E}_2$ -part of (5.3), measuring the gravitational charge, does not occur in  $\Delta_{(TT)}$ .

## 6. Conclusion

We have calculated the one-loop threshold corrections to the gauge couplings of  $U(1)$  gauge bosons which arise from heterotic  $N=2$ ,  $d = 4$  compactifications on a torus. These results fit into the framework of the underlying  $N=2$  supergravity theory. Using (4.21) and [18] we may cast the effective gauge couplings (3.12) into the final form:

$$-\frac{\Delta_{(TU)}}{32\pi^2} = \frac{1}{2}\text{Re}\{D_T D_U f\} = \frac{1}{16\pi^2} \left[ \ln |j(T) - j(U)|^2 - G^{(1)} + \sigma(T, U) \right] , \quad (6.1)$$

where  $\sigma(T, U)$  are the universal one-loop corrections appearing in all gauge threshold corrections of heterotic  $N=2$  theories [18]. The correction  $G^{(1)}$  describes the mixing of the dilaton and the moduli fields at one-loop [6][12][18].

In section 4 we have calculated several world-sheet  $\tau$ -integrals as they appear in string amplitude calculations from various contractions of the internal bosonic coordinate fields. These expressions appear quite general in heterotic torus compactifications from  $N=1$  in  $d = 10, 6, 4$  to  $d = 8, 4, 2$ . The relevant string amplitudes take a generic form, given by a  $\tau$ -integral over the (new) supersymmetric index (4.2) (or variants of it depending on  $d$ ), completed with momentum insertions of internal bosonic fields, which take into account either vertex operator contractions or charge insertions. In the case of  $K3 \times T^2$  compactifications, a part of the supersymmetric index (4.2) is related to a modular function  $f$ , which is the  $N=2$ ,  $d = 4$  prepotential. Many  $\tau$ -integrals can be expressed through  $f$  and its derivatives. Such relations between string-amplitudes and a function  $f$  and its derivatives, as established here for  $N=2$  in  $d = 4$ , should also exist in any torus compactifications of e.g.  $d = 10, 4$  heterotic string theory.

**Acknowledgments:** We are very grateful to I. Antoniadis, J.-P. Derendinger and W. Lerche for important discussions. K.F. thanks the Institut de Physique Théorique de l'Université de Neuchâtel for the friendly hospitality. This work is supported by the Swiss National Science Foundation, and the European Commission TMR programme ERBFMRX-CT96-0045, in which K.F. is associated to HU Berlin and St. St. to OFES no. 95.0856.

## Appendix A. Poisson resummation

In this section we want to perform a Poisson transformation on the expression

$$\mathcal{S} = \sum_{\vec{p} \in \Lambda^*} e^{-\pi \vec{p}^t \alpha \vec{p}} e^{2\pi i \vec{y}^t \vec{p}} (\vec{p}^t A \vec{p} + \vec{b}^t \vec{p} + a_0) (\vec{p}^t D \vec{p} + \vec{c}^t \vec{p} + e_0) , \quad (\text{A.1})$$

for some matrices  $\alpha, A, D$  ( $\det \alpha \neq 0$ ), vectors  $\vec{y}, \vec{b}, \vec{c}$  and scalars  $a_0, e_0$ . This is achieved like one does a Fourier transformation on a periodic function  $F(\vec{x})$ , i.e.  $[F(\vec{x} + \vec{p}_0) = F(\vec{x})]$ :

$$F(\vec{x}) = \sum_{\vec{p} \in \Lambda} e^{-\pi (\vec{p} + \vec{x})^t \alpha (\vec{p} + \vec{x})} e^{2\pi i \vec{y}^t (\vec{p} + \vec{x})} . \quad (\text{A.2})$$

With

$$F^*(\vec{q}) = \frac{1}{\text{vol}(\Lambda)} \int_{-\infty}^{\infty} d\vec{x} e^{-2\pi i \vec{q}^t \vec{x}} F(\vec{x}) , \quad (\text{A.3})$$

we may write:

$$\begin{aligned} F(\vec{x}) &= \sum_{\vec{q} \in \Lambda^*} e^{2\pi i \vec{q}^t \vec{x}} F^*(\vec{q}) \\ &= \frac{1}{\sqrt{\det \alpha}} \frac{1}{\text{vol}(\Lambda)} \sum_{\vec{q} \in \Lambda^*} e^{-2\pi i \vec{q}^t \vec{x}} e^{-\pi (\vec{y} + \vec{q})^t \alpha^{-1} (\vec{y} + \vec{q})} . \end{aligned} \quad (\text{A.4})$$

Along that way a Fourier transformation on (A.1) yields

$$\begin{aligned} \mathcal{S} &= \frac{1}{\sqrt{\det \alpha}} \frac{1}{\text{vol}(\Lambda^*)} \sum_{\vec{q} \in \Lambda} e^{-\pi (\vec{y} + \vec{q})^t \alpha^{-1} (\vec{y} + \vec{q})} \left\{ a'_0 e'_0 + \frac{1}{2\pi} \left[ e'_0 \text{Tr}(\alpha^{-1} A) + a'_0 \text{Tr}(\alpha^{-1} D) \right. \right. \\ &\quad \left. \left. + \left( \vec{b} + i(A^t + A)\alpha^{-1}(\vec{y} + \vec{q}) \right)^t \alpha^{-1} \left( \vec{c} + i(D^t + D)\alpha^{-1}(\vec{y} + \vec{q}) \right) \right] \right. \\ &\quad \left. + \frac{1}{4\pi^2} \left[ \text{Tr}(\alpha^{-1} A) \text{Tr}(\alpha^{-1} D) + \text{Tr}(\alpha^{-1} A \alpha^{-1} D^t) + \text{Tr}(\alpha^{-1} A \alpha^{-1} D) \right] \right\} \end{aligned} \quad (\text{A.5})$$

with the following abbreviations:

$$\begin{aligned} a'_0 &= a_0 - (\vec{y} + \vec{q})^t \alpha^* A \alpha^{-1} (\vec{y} + \vec{q}) + i \vec{b}^t \alpha^{-1} (\vec{y} + \vec{q}) \\ e'_0 &= e_0 - (\vec{y} + \vec{q})^t \alpha^* D \alpha^{-1} (\vec{y} + \vec{q}) + i \vec{c}^t \alpha^{-1} (\vec{y} + \vec{q}) . \end{aligned} \quad (\text{A.6})$$

## Appendix B. Integrals

### B.1. Orbit $I_1$

For the orbit  $I_1$  we have to face the following integrals

$$I_1^{\alpha, \beta, n} := \sum_{k, j, p} \tilde{I}_1^{\alpha, \beta, n} = T_2 e^{-2\pi i \bar{T} k p} \int_{H_+} \frac{d\tau}{\tau_2^{2+\beta}} \sum_{k, j, p} \tau_1^\alpha e^{-\frac{\pi T_2}{\tau_2 U_2} |k\tau - j - pU|^2} e^{-2\pi i \bar{\tau} n} , \quad (\text{B.1})$$

for the cases  $\alpha = 0, \dots, 4$  and  $\beta = -1, 0, \dots, 3$ . Clearly, the case  $\alpha = 0$  and  $\beta = 0$  corresponds to the integral performed in [16]. The case  $\beta = -1$  is needed for the integrals appearing in sect. 4.2. We expanded the anti-holomorphic function  $\overline{F}(\overline{\tau})$  in a power series:

$$\overline{F}(\overline{\tau}) = \sum_{n \geq -1} c_n e^{-2\pi i \overline{\tau} n} . \quad (\text{B.2})$$

The  $\tau_1$ -integration of (B.1) can be reduced to Gaussian integrals:

$$\begin{aligned} \tilde{I}_1^{\alpha, \beta, n} &= \frac{k^{-\alpha}}{k} \sqrt{T_2 U_2} e^{-2\pi i \overline{T} k p} e^{2\pi T_2 k p} e^{-2\pi i \frac{n}{k} (j + p U_1)} \\ &\times \int_0^\infty \frac{d\tau_2}{\tau_2^{\frac{3}{2} + \beta}} e^{-\frac{\pi T_2}{U_2} (k + \frac{n U_2}{k T_2})^2 \tau_2} e^{-\pi p^2 T_2 U_2 / \tau_2} \mathcal{X}_\alpha , \end{aligned} \quad (\text{B.3})$$

with:

$$\begin{aligned} \mathcal{X}_0 &= 1 \\ \mathcal{X}_1 &= -i \frac{n}{k} \frac{\tau_2 U_2}{T_2} + j + p U_1 \\ \mathcal{X}_2 &= \frac{1}{2\pi} \frac{\tau_2 U_2}{T_2} + (-i \frac{n}{k} \frac{\tau_2 U_2}{T_2} + j + p U_1)^2 \\ \mathcal{X}_3 &= \frac{3}{2\pi} \frac{\tau_2 U_2}{T_2} (-i \frac{n}{k} \frac{\tau_2 U_2}{T_2} + j + p U_1) + (-i \frac{n}{k} \frac{\tau_2 U_2}{T_2} + j + p U_1)^3 \\ \mathcal{X}_4 &= \frac{3}{4\pi^2} \left( \frac{\tau_2 U_2}{T_2} \right)^2 + \frac{3}{\pi} \frac{\tau_2 U_2}{T_2} (-i \frac{n}{k} \frac{\tau_2 U_2}{T_2} + j + p U_1)^2 + (-i \frac{n}{k} \frac{\tau_2 U_2}{T_2} + j + p U_1)^4 . \end{aligned} \quad (\text{B.4})$$

Next, we have to do the  $\tau_2$ -integration. For  $\beta = 0, 1, 2$  we may borrow results<sup>6</sup> from [26] for the integrals

$$\tilde{I}_1^{0, \beta, n} = \left( \frac{k}{|p| U_2} \right)^\beta \tilde{I}_1 \times \begin{cases} \left[ 1 + n \frac{\mathcal{U}_2}{T_2} \right]^{-1} & \beta = -1 , \\ 1 & \beta = 0 , \\ \left[ 1 + \frac{1}{T_2} (n \mathcal{U}_2 + \frac{1}{2\pi}) \right] & \beta = 1 , \\ \left[ 1 + \frac{1}{T_2} (2n \mathcal{U}_2 + \frac{3}{2\pi}) + \frac{1}{T_2^2} (n^2 \mathcal{U}_2^2 + \frac{3n \mathcal{U}_2}{2\pi} + \frac{3}{4\pi^2}) \right] & \beta = 2 , \\ \left[ 1 + \frac{1}{T_2} (3n \mathcal{U}_2 + \frac{3}{\pi}) + \frac{1}{T_2^2} (3n^2 \mathcal{U}_2^2 + \frac{6n \mathcal{U}_2}{\pi} + \frac{15}{4\pi^2}) \right. \\ \quad \left. + \frac{1}{T_2^3} (n^3 \mathcal{U}_2^3 + \frac{3n^2}{\pi} \mathcal{U}_2^2 + \frac{15}{4\pi^2} n \mathcal{U}_2 + \frac{15}{8\pi^3}) \right] & \beta = 3 , \end{cases} \quad (\text{B.5})$$

---

<sup>6</sup> We thank N.A. Obers for explanation of the notation of [26]. In their final formulae one must replace  $p \rightarrow |p|$ .

with

$$\tilde{I}_1 = \frac{1}{k|p|} e^{-2\pi i \overline{T} k p} e^{-2\pi i \frac{n}{k} (j+pU_1-i|p|U_2)} e^{2\pi T_2 k (p-|p|)} , \quad (\text{B.6})$$

and:

$$\mathcal{T}_2 = k|p|T_2 \quad , \quad \mathcal{U}_2 = \frac{|p|U_2}{k} . \quad (\text{B.7})$$

Before we continue, let us recover

$$\begin{aligned} I_1^{0,0,n} &= \sum_{\substack{k>0 \\ l \in \mathbf{Z}}} \delta_{n,kl} \mathcal{L}i_1 \left[ e^{2\pi i (kT+lU)} \right] + hc. , \\ I_1^{0,1,n} &= \sum_{\substack{k>0 \\ l \in \mathbf{Z}}} \delta_{n,kl} \mathcal{P} \left[ e^{2\pi i (kT+lU)} \right] + hc. , \end{aligned} \quad (\text{B.8})$$

with:

$$\mathcal{P}(kT + lU) = \text{Im}(kT + lU) \mathcal{L}i_2 \left[ e^{2\pi i (kT+lU)} \right] + \frac{1}{2\pi} \mathcal{L}i_3 \left[ e^{2\pi i (kT+lU)} \right] . \quad (\text{B.9})$$

Finally for the cases of interest, we have reduced everything to integrals  $\tilde{I}_1^{0,\beta,n}$  given in eq. (B.5).

$$\begin{aligned} k\tilde{I}_1^{1,\beta,n} &= -i \frac{n}{k} \frac{U_2}{T_2} \tilde{I}_1^{0,\beta-1,n} + (j+pU_1) \tilde{I}_1^{0,\beta,n} \\ k^2 \tilde{I}_1^{2,\beta,n} &= -\frac{n^2}{k^2} \frac{U_2^2}{T_2^2} \tilde{I}_1^{0,\beta-2,n} + \frac{1}{2\pi} \frac{U_2}{T_2} \tilde{I}_1^{0,\beta-1,n} - 2i(j+pU_1) \frac{n}{k} \frac{U_2}{T_2} \tilde{I}_1^{0,\beta-1,n} \\ &\quad + (j+pU_1)^2 \tilde{I}_1^{0,\beta,n} \\ k^3 \tilde{I}_1^{3,\beta,n} &= i \frac{n^3}{k^3} \frac{U_2^3}{T_2^3} \tilde{I}_1^{0,\beta-3,n} - \frac{3}{2\pi} i \frac{n}{k} \frac{U_2^2}{T_2^2} \tilde{I}_1^{0,\beta-2,n} - 3 \frac{n^2}{k^2} (j+pU_1) \frac{U_2^2}{T_2^2} \tilde{I}_1^{0,\beta-2,n} \\ &\quad + \frac{3}{2\pi} (j+pU_1) \frac{U_2}{T_2} \tilde{I}_1^{0,\beta-1,n} - 3i(j+pU_1)^2 \frac{n}{k} \frac{U_2}{T_2} \tilde{I}_1^{0,\beta-1,n} + (j+pU_1)^3 \tilde{I}_1^{0,\beta,n} \\ k^4 \tilde{I}_1^{4,\beta,n} &= \frac{n^4}{k^4} \frac{U_2^4}{T_2^4} \tilde{I}_1^{0,\beta-4,n} - \frac{3}{\pi} \frac{n^2}{k^2} \frac{U_2^3}{T_2^3} \tilde{I}_1^{0,\beta-3,n} + 4i \frac{n^3}{k^3} \frac{U_2^3}{T_2^3} (j+pU_1) \tilde{I}_1^{0,\beta-3,n} \\ &\quad - \frac{6}{\pi} i \frac{n}{k} (j+pU_1) \frac{U_2^2}{T_2^2} \tilde{I}_1^{0,\beta-2,n} + \frac{3}{4\pi^2} \frac{U_2^2}{T_2^2} \tilde{I}_1^{0,\beta-2,n} - 6 \frac{n^2}{k^2} \frac{U_2^2}{T_2^2} (j+pU_1)^2 \tilde{I}_1^{0,\beta-2,n} \\ &\quad + \frac{3}{\pi} \frac{U_2}{T_2} (j+pU_1)^2 \tilde{I}_1^{0,\beta-1,n} - 4i \frac{n}{k} \frac{U_2}{T_2} (j+pU_1)^3 \tilde{I}_1^{0,\beta-1,n} + (j+pU_1)^4 \tilde{I}_1^{0,\beta,n} . \end{aligned} \quad (\text{B.10})$$

Now, the important and nice fact is, that after expanding the expression (B.10), the  $j$ -sum, in the combination of (4.10), (4.16), (4.25), (4.32) and (4.34) becomes trivial and gives the

restriction  $n = kl$ ,  $l \in \mathbf{Z}$ . Then, after a straightforward calculation the orbits  $I_1^{\mathcal{I}_{w_T, w_U}}$  belonging to the integral  $\mathcal{I}_{w_T, w_U}$  can be determined:

$$\begin{aligned}
I_1^{\mathcal{I}_{2,-2}} &= - \sum_{\substack{k>0 \\ l \in \mathbf{Z}}} \sum_{p>0} \delta_{n,kl} \left( 2\frac{k^2}{p} + \frac{k}{\pi T_2 p^2} + \frac{1}{4\pi^2 T_2^2 p^3} \right) x^p \\
&\quad - \sum_{\substack{k>0 \\ l \in \mathbf{Z}}} \sum_{p>0} \delta_{n,kl} \left( 2\frac{U_2^2 l^2}{T_2^2 p} + \frac{U_2 l}{\pi T_2^2 p^2} + \frac{1}{4\pi^2 T_2^2 p^3} \right) \bar{x}^p \\
I_1^{\mathcal{I}_{0,0}} &= - \sum_{\substack{k>0 \\ l \in \mathbf{Z}}} \sum_{p>0} \delta_{n,kl} \left( \frac{2kl}{p} + \frac{l}{\pi T_2 p^2} + \frac{k}{\pi U_2 p^2} + \frac{1}{2\pi^2 T_2 U_2 p^3} \right) x^p + hc. \\
I_1^{\mathcal{I}_{4,-2}} &= -2i \sum_{\substack{k>0 \\ l \in \mathbf{Z}}} \sum_{p>0} \delta_{n,kl} k^3 x^p \\
I_1^{\mathcal{I}_{2,0}} &= - \sum_{\substack{k>0 \\ l \in \mathbf{Z}}} \sum_{p>0} \delta_{n,kl} \left( 4i\pi k^2 l + \frac{2ik^2}{U_2 p} + \frac{ik}{\pi T_2 U_2 p^2} + \frac{2ikl}{T_2 p} + \frac{i}{\pi^2 T_2^2 U_2 p^3} + \frac{il}{\pi T_2^2 p^2} \right) x^p \\
&\quad - \sum_{\substack{k>0 \\ l \in \mathbf{Z}}} \sum_{p>0} \delta_{n,kl} \left( \frac{i}{4\pi^2 T_2^2 U_2 p^3} - \frac{il}{2\pi T_2^2 p^2} \right) \bar{x}^p,
\end{aligned} \tag{B.11}$$

with  $x := e^{2\pi i(kT+lU)}$ ,  $\bar{x} := e^{-2\pi i(k\bar{T}+\bar{l}\bar{U})}$ .

## B.2. Orbit $I_2$

For the orbit  $I_2$  the following integrals appear

$$I_2^{\alpha, \beta, \gamma, n} = T_2 \int_{-\frac{1}{2}}^{+\frac{1}{2}} d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2^{2+\gamma}} \sum'_{(j,p)} j^\alpha p^\beta e^{-\frac{\pi T_2}{\tau_2 U_2} |j+Up|^2} e^{-2\pi i \bar{\tau} n}. \tag{B.12}$$

Here the prime at the sum means that we do not sum over  $(j, p) = (0, 0)$ , which is taken into account in  $I_0$ . The case  $\gamma = 0$  describes the respective integral of [16]. In that case one has to regularize the integral. We will only need cases with  $\gamma \neq 0$ . Therefore we have not to introduce an IR-regulator. See also [27] for discussions. This also means that our results will not produce any non-harmonic  $\ln(T_2 U_2)$ -pieces. Such terms are usually signals of potential anomalies arising in the IR. The  $\tau_1$ -integration is trivial and simply projects onto massless states, i.e.  $n = 0$ . For the  $\tau_2$ -integration we again may use results of [26]:

$$\int_0^\infty \frac{d\tau_2}{\tau_2^{2+\gamma}} \sum'_{(j,p)} e^{-\frac{\pi T_2}{\tau_2 U_2} |j+Up|^2} = \Gamma(\gamma + 1) \left( \frac{U_2}{\pi T_2} \right)^{\gamma+1} \sum'_{(j,p)} \frac{1}{|j + pU|^{2\gamma+2}}. \tag{B.13}$$

Therefore, we only have to concentrate on the sum

$$\sum'_{(j,p)} \frac{j^\alpha p^\beta}{|j + Up|^{2\gamma+2}} . \quad (\text{B.14})$$

Let us first perform the summation for the case  $p \neq 0$  and write:

$$\begin{aligned} \sum_{p \neq 0} \frac{j^\alpha p^\beta}{[(j + U_1 p)^2 + p^2 U_2^2]^{\gamma+1}} &= \frac{1}{\gamma!} \sum_{p > 0} \left( \frac{1}{i} \frac{\partial}{\partial \theta} \right)^\alpha p^{\beta-2\gamma} \left( \frac{(-1)}{2U_2} \frac{\partial}{\partial U_2} \right)^\gamma \\ &\times \sum_{j=-\infty}^{\infty} \left[ \frac{e^{i\theta j}}{(j + U_1 p)^2 + p^2 U_2^2} + \frac{(-1)^\beta e^{i\theta j}}{(j - U_1 p)^2 + p^2 U_2^2} \right] \Big|_{\theta=0} . \end{aligned} \quad (\text{B.15})$$

After using a Sommerfeld–Watson transformation, introduced in [10], ( $C > 0$ ,  $0 \leq \theta \leq 2\pi$ )

$$\sum_{j=-\infty}^{\infty} \frac{e^{i\theta j}}{(j + B)^2 + C^2} = \frac{\pi}{C} e^{-i\theta(B-iC)} \frac{1}{1 - e^{-2\pi i(B-iC)}} + \frac{\pi}{C} e^{-i\theta(B+iC)} \frac{e^{2\pi i(B+iC)}}{1 - e^{-2\pi i(B+iC)}} , \quad (\text{B.16})$$

we end up with

$$\begin{aligned} &\frac{\pi}{\gamma!} \left( \frac{-1}{2U_2} \frac{\partial}{\partial U_2} \right)^\gamma \left\{ \frac{1}{U_2} \left[ [(-U)^\alpha + (-1)^\beta U^\alpha] \sum_{l>0} \mathcal{L}_{i_{1-\alpha-\beta+2\gamma}}(q_U^l) \right. \right. \\ &\quad + [(-\bar{U})^\alpha + (-1)^\beta \bar{U}^\alpha] \sum_{l>0} \mathcal{L}_{i_{1-\alpha-\beta+2\gamma}}(\bar{q}_U^l) \\ &\quad \left. \left. + [(-\bar{U})^\alpha + (-1)^\beta U^\alpha] \zeta(1 - \alpha - \beta + 2\gamma) \right] \right\} . \end{aligned} \quad (\text{B.17})$$

Let us now come to the case  $p = 0$  of (B.14)

$$Q^{\alpha,\beta,\gamma} := \delta_{\beta,0} [1 + (-1)^\alpha] \sum_{j=1}^{\infty} \frac{1}{j^{2\gamma+2-\alpha}} , \quad (\text{B.18})$$

which only gives a non-zero contribution for  $\beta = 0$ . Moreover, we must only consider  $\alpha \in 2\mathbf{Z}$ . For the examples we discuss in section 4.1. we have  $\alpha + \beta = \gamma = 2$  and for the cases in section 4.2,  $\alpha + \beta = 4$ ,  $\gamma = 3$ . In both cases, the sum (B.18) becomes:

$$Q^{2,0,2} = Q^{4,0,3} = 2 \sum_{j=1}^{\infty} \frac{1}{j^4} = \frac{\pi^4}{45} . \quad (\text{B.19})$$

Putting everything together, we obtain for  $I_2^{\alpha,\beta,\gamma,n}$ :

$$\begin{aligned}
I_2^{\alpha,\beta,\gamma,n} &= \delta_{n,0} T_2 \Gamma(\gamma+1) \left( \frac{U_2}{\pi T_2} \right)^{\gamma+1} \\
&\times \left\{ \frac{\pi}{\gamma!} \left( \frac{-1}{2U_2} \frac{\partial}{\partial U_2} \right)^\gamma \left[ \frac{1}{U_2} \left( [(-U)^\alpha + (-1)^\beta U^\alpha] \sum_{l>0} \mathcal{L}i_{1-\alpha-\beta+2\gamma}(q_U^l) \right. \right. \right. \\
&\quad \left. \left. + [(-\bar{U})^\alpha + (-1)^\beta \bar{U}^\alpha] \sum_{l>0} \mathcal{L}i_{1-\alpha-\beta+2\gamma}(\bar{q}_U^l) \right. \right. \\
&\quad \left. \left. + [(-\bar{U})^\alpha + (-1)^\beta U^\alpha] \zeta(1-\alpha-\beta+2\gamma) \right) \right] \\
&\quad \left. + Q^{\alpha,\beta,\gamma} \right\} .
\end{aligned} \tag{B.20}$$

## References

- [1] B. de Wit, P.G. Lauwers, R. Philippe, S.Q. Su, A. van Proeyen, *Phys. Lett.* **B 134** (1984) 37; B. de Wit, A. van Proeyen, *Nucl. Phys.* **B 245** (1984) 89; B. de Wit, P.G. Lauwers, A. van Proeyen, *Nucl. Phys.* **B 255** (1985) 569; E. Cremmer, C. Kounnas, A. van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit, L. Girardello, *Nucl. Phys.* **B 250** (1985) 385
- [2] A. Strominger, *Comm. Math. Phys.* **133** (1990) 163
- [3] L. Dixon, V. Kaplunovsky and J. Louis, *Nucl. Phys.* **B 329** (1990) 27
- [4] M. Dine, P. Huet and N. Seiberg, *Nucl. Phys.* **B 322** (1989) 301; L. Ibanez, W. Lerche, D. Lüst and S. Theisen, *Nucl. Phys.* **B 352** (1991) 435
- [5] P. Mayr and S. Stieberger, *Phys. Lett.* **B 355** (1995) 107
- [6] *see e.g.:* J.-P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, *Nucl. Phys.* **B 372** (1992) 145; V. Kaplunovsky and J. Louis, *Nucl. Phys.* **B 444** (1995) 191
- [7] S. Kachru and C. Vafa, *Nucl. Phys.* **B 450** (1995) 69
- [8] A. Klemm, W. Lerche and P. Mayr, *Phys. Lett.* **B 357** (1995) 313
- [9] V. Kaplunovsky, J. Louis and S. Theisen, *Phys. Lett.* **B 357** (1995) 71
- [10] J.A. Harvey and G. Moore, *Nucl. Phys.* **B 463** (1996) 315
- [11] A. Ceresole, R. d' Auria, S. Ferrara and A. van Proeyen, *Nucl. Phys.* **B 444** (1995) 92
- [12] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Nucl. Phys.* **B 451** (1995) 53
- [13] I. Antoniadis, S. Ferrara, E. Gava, K. Narain and T. Taylor, *Nucl. Phys.* **B 447** (1995) 35
- [14] J. Lauer, D. Lüst, S. Theisen, *Nucl. Phys.* **B 309** (1988) 771
- [15] V. Kaplunovsky, *Nucl. Phys.* **B 307** (1988) 145
- [16] L. Dixon, V. Kaplunovsky and J. Louis, *Nucl. Phys.* **B 355** (1991) 649
- [17] J.P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, *Nucl. Phys.* **B 372** (1992) 145; I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Nucl. Phys.* **B 407** (1993) 706
- [18] H.P. Nilles and S. Stieberger, *Nucl. Phys.* **B 499** (1997) 3
- [19] J. Ellis, P. Jetzer, L. Mizrachi, *Nucl. Phys.* **B 303** (1988) 1
- [20] J. A. Minahan, *Nucl. Phys.* **B 298** (1988) 36; P. Mayr, S. Stieberger, *Nucl. Phys.* **B 412** (1994) 502; K. Förger, B.A. Ovrut, S. Theisen, D. Waldram, *Phys. Lett.* **B 388** (1996) 512
- [21] S. Stieberger, NEIP-001/97, work in progress
- [22] R.E. Borcherds, *Invent. Math.* **120** (1995) 161
- [23] W. Lerche, B.E.W. Nilsson and A.N. Schellekens, *Nucl. Phys.* **B 289** (1987) 609; W. Lerche, B.E.W. Nilsson, A.N. Schellekens and N.P. Warner, *Nucl. Phys.* **B 299** (1988) 91; W. Lerche, A.N. Schellekens and N.P. Warner, *Phys. Rept.* **177** (1989) 1

- [24] W. Lerche, *Nucl. Phys.* **B 308** (1988) 102
- [25] I. Antoniadis, E. Gava and K.S. Narain, *Nucl. Phys.* **B 383** (1992) 92; *Phys. Lett.* **B 283** (1992) 209
- [26] C. Bachas, C. Fabre, E. Kiritsis, N.A. Obers and P. Vanhove, hep-th/9707126
- [27] E. Kiritsis and C. Kounnas, *Nucl. Phys.* **B 442** (1995) 472; P.M. Petropoulos and J. Rizos, *Phys. Lett.* **B 374** (1996) 49